Permutations and Combinations

↳ Factorial n ($n!$) or $\lfloor n$ for any natural number is defined as the product of first n natural numbers.

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-2)(n-1)n$$

$$n! = (n-1)!\, n = (n-2)!\,(n-1)(n) = \cdots$$

$$0! = 1 \quad (\text{Convention}).$$

↳ If $x! = y!$, then either $x = y$ or $x = 0$ and $y = 1$ or $x = 1$ and $y = 0$

↳ Exponent of prime p in $n!$ is equal to $= \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots$

↳ Number of zeros at the end of $n!$ is simply $\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + \cdots$

$\boxed{\text{Fundamental Principle of Counting}}$

Addition Rule: If two mutually exclusive events A and B can occur in m and n ways respectively then the total number of ways in which A or B can happen is $m+n$.

Multiplication Rule: If two independent events A and B can occur in m and n ways respectively then the total number of ways in which A and B can occur is $m \cdot n$.

Combinations: Arrangements in which order does NOT matter

↳ Total number of ways to select r things from n distinct things

$$= {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{r!(n-r)!}$$

Properties: i) ${}^nC_0 = {}^nC_n = 1$

ii) ${}^nC_r = {}^nC_k \Rightarrow r = k$ or $r = n-k$

iii) ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$

iv) $r \cdot {}^nC_r = n \cdot {}^{n-1}C_{r-1}$

v) ${}^nC_r$ is maximum at $r = \frac{n}{2}$ when n is even and $r = \frac{n \pm 1}{2}$ when n is odd.

↳ Total number of ways to select r objects from n distinct objects such that p particular objects are always included in the selection is

$$^{n-p}C_{r-p}$$

↳ Total number of ways to arrange r objects from n distinct objects such that p particular objects are always included in the arrangement is

$$^{n-p}C_{r-p} \; r!$$

↳ Total number of ways to select r objects from n distinct objects such that p particular objects are always excluded is

$^{n-p}C_r$

↳ Total number of ways to arrange r objects from n distinct objects such that p particular objects are always excluded in the arrangement is:

$$^{n-p}C_r \cdot r!$$

↳ The total number of ways to arrange n objects such that p particular objects always remain together in the arrangement is:

$$(n-p+1)!\ p!$$

↳ The total number of ways to arrange n objects such that p particular objects are always seperated is:

$$^{n-p+1}C_p \cdot p!\ (n-p)!$$

↳ In problems pertaining to "at least" or "at most" phrases,

a) make cases

OR

b) use Principle of Inclusion and Exclusion.

c) Don't use multiplication rule or at least check its validity.

↳ Selection of r objects from n when not all the objects are distinct.

- Make cases

↳ Selection of one or more object(s) when |

a) Total number of ways to select any number of objects from n distinct objects is 2^n.

b) Total number of ways to select at least one object from n distinct objects is $2^n - 1$.

Selection from identical objects:

↳ The total number of way(s) to select r objects from n identical objects is 1.

↳ The total number of ways to select any number of objects from n identical objects is $n+1$.

c) The total number of ways to select at least one object from n identical objects is n.

$\boxed{\hookrightarrow \text{ Given a number } N = p_1^{a} p_2^{b} p_3^{c} \ldots \text{ where } p_i \text{ are primes}}$

a) Number of divisors $= (a+1)(b+1)(c+1)\ldots$

c) Sum of divisors $= \left(\dfrac{p_1^{a+1}-1}{p_1-1}\right)\cdot\left(\dfrac{p_2^{b+1}-1}{p_2-1}\right)\left(\dfrac{p_3^{c+1}-1}{p_3-1}\right)\ldots$

$\hookrightarrow$ Total number of subsets of a set containing n element is 2^n.

DIVISION INTO Groups (objects distinct)

↳ Total number of ways to divide n distinct things into groups containing $m, p, q, \ldots$ (Groups are distinct).

$$= \frac{n!}{m!\,p!\,q!\ldots}$$

↳ Total number of ways to distribute mn objects into m groups such that each group has equal number of objects (groups are distinct)

$$= \frac{(mn)!}{(n!)^m}$$

↳ Total number of ways to distribute mn objects into m groups such that each group has equal number of objects (groups are identical)

$$= \frac{(mn)!}{m!(n!)^m}$$

DIVISION into group (Objects identical)

↳ The total number of ways to distribute n identical objects into r distinct group such that each group gets at least one object = coefficient of x^n in $(x+x^2+x^3+\dots)^r$

$$= {}^{n-1}C_{r-1}$$

↳ The total number of ways to distribute n identical objects into r distinct group such that each group gets any number of object(s)

coefficient of x^n in $(1+x+x^2+\cdots)^r$

$$= {}^{n+r-1}C_{r-1}$$

Principle of Inclusion:

↳ Total number of derangements of n is equal to

$$n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!} \qquad \frac{(-1)^n}{n!}\right]$$

↳ Total number of ways to distribute n distinct objects into r distinct groups such that each group gets at least one object is:

$$r^n - {}^rC_1 (r-1)^n + {}^rC_2 (r-2)^n - \dots$$

Circular Permutations:

↳ Total number of ways to arrange n distinct things around a circle is $(n-1)!$

↳ If the clockwise arrangement and counter-clockwise arrangements are same then the total number of distinct arrangements is equal to $\frac{(n-1)!}{2}$.

Geometrical Combinatorics

↳ For n points in a plane the maximum number of lines is ${}^nC_2$, the maximum number of triangles is ${}^nC_3$

↳ In an n sided convex polygon the total number of diagonals is ${}^nC_2 - n$.